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NOTE

ON THE FACTORIZATION OF GRAPHS WITH EXACTLY ONE VERTEX OF INFINITE DEGREE*

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We give a necessary and sufficient condition for the existence of a 1-factor in graphs with exactly one vertex of infinite degree.

1. Introduction

The following well-known necessary and sufficient condition for the existence of a 1-factor in locally finite graphs is due to Tutte [5]:

Theorem A. *A locally finite graph $G=(V,E)$ has a 1-factor if and only if $C_1(V\setminus S)\leqslant|S|$ for all finite subset S of V.* (See notations below.)

In the present note, we extend this theorem to graphs with exactly one vertex of infinite degree. For bipartite graphs with exactly one vertex of infinite degree, our result reduces to a theorem due to Jung and Rado [4].

2. Notations and terminology

Graphs considered in this note are undirected without loops or multiple edges.

Let $G=(V,E)$ be a graph. A 1-*factor*, or *perfect matching*, of G is a set of pairwise disjoint edges of G containing all vertices. We say that G is *factorizable* if it contains at least one 1-factor.

A finite graph is 1-*factor critical* if by deleting any vertex one obtains a factorizable graph. A 1-factor critical graph has clearly an odd number of vertices.

We denote by $C_1(G)$ the number of connected components with odd cardinalities of G, and by $C_{cr}(G)$ the number of connected components of G which are 1-factor critical.

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Given a subset S of V, we denote by $G[S]$ the subgraph of G induced by S. If no confusion results we abbreviate $C_1(G[S])$ and $C_{cr}(G[S])$ to $C_1(S)$ and $C_{cr}(S)$ respectively.

Given a vertex v, we denote by $A(v)$ the set of vertices adjacent to v in G.

A graph is *locally finite* if $A(v)$ is finite for every vertex v.

3. Statement of the results

Theorem 3.1. *A graph $G=(V,E)$ with exactly one vertex v_0 of infinite degree is factorizable if and only if*

(1.1) $C_1(V\setminus S)\leqslant|S|$ *for all finite subsets S of V,*

(1.2) $A(v_0)\nsubseteq\bigcup\{S\subseteq V\setminus\{v_0\}$: *S finite*, $C_1(V\setminus[S\cup\{v_0\}])=|S|\}$.

Corollary 3.2. *A graph $G=(V,E)$ with exactly one vertex v_0 of infinite degree is factorizable if and only if*

(2.1) $C_{cr}(V\setminus S)\leqslant|S|$ *for all finite subsets S of V,*

(2.2) $A(v_0)\nsubseteq\bigcup\{S\subseteq V\setminus\{v_0\}$: *S finite*, $C_{cr}(V\setminus[S\cup\{v_0\}])=|S|\}$.

The following lemma proved in [1] is needed to prove Theorem 3.1 and Corollary 3.2:

Lemma 3.3. *Let $G=(V,E)$ be a locally finite graph and k a non-negative integer. If there exists a finite subset S of V such that $C_1(V\setminus S)\geqslant|S|+k$, then there exists a finite subset T of V such that $S\subseteq T$ and $C_{cr}(V\setminus T)\geqslant|T|+k$.*

From Lemma 3.3 a strengthening of Theorem A [1] follows:

Theorem B. *A locally finite graph $G=(V,E)$ is factorizable if and only if $C_{cr}(V\setminus S)\leqslant|S|$ for all finite subsets S of V.*

In the finite case, Theorem B is a well-known result, however we have been unable to find an explicit reference in the literature. The papers [2] and [3] can be given as implicit references.

4. Proof

Let $G=(V,E)$ be a graph with exactly one vertex v_0 of infinite degree.

(1) *If condition* (1.1) *holds for G and if G is not factorizable, then $G[V\setminus\{v_0\}]$ is factorizable.*

Since $G[V\backslash\{v_0\}]$ is locally finite, from Theorem A it is enough to prove that $C_1(V\backslash[S\cup\{v_0\}])\leqslant|S|$ for all finite subset S of $V\backslash\{v_0\}$. Assume that there is a finite subset S of $V\backslash\{v_0\}$ such that

$$C_1(V\backslash[S\cup\{v_0\}])\geqslant|S|+1.$$

Since (1.1) holds for G we have $C_1(V\backslash[S\cup\{v_0\}])=|S|+1$. By Lemma 3.3 there is a finite subset T of V such that $S\cup\{v_0\}\subseteq T$ and $C_{cr}(V\backslash T)\geqslant|T|$. Since (1.1) holds for G we have $C_{cr}(V\backslash T)=|T|$, and every connected components of $G[V\backslash T]$ with odd cardinality is 1-factor critical.

On the other hand we prove that every connected component of $G[V\backslash T]$ with even or infinite cardinality is factorizable. Let C be such a component of $G[V\backslash T]$. Since v_0 belongs to T, $G[C]$ is locally finite. If $G[C]$ is not factorizable, from Theorem A there is a finite subset U of C such that $C_1(C\backslash U)\geqslant|U|+1$. Therefore we have

$$C_1(V\backslash[T\cup U])=C_1(V\backslash T)+C_1(G[C\backslash U])\geqslant|S\cup T|+1,$$

contradicting (1.1).

Since every connected component of $G[V\backslash T]$ with odd cardinality is 1-factor critical, the subgraph of G induced by T and the components of $G[V\backslash T]$ with odd cardinalities have a 1-factor. This 1-factor can be extended to a 1-factor of G, since the connected components of $G[V\backslash T]$ with even or infinite cardinalities are factorizable. The contradiction follows from the hypothesis that G is not factorizable, achieving the proof of (1).

(2) *If* (1.1) *holds for G and if G is not factorizable, then* (1.2) *does not hold for G.*

Let y be a vertex of $A(v_0)$. Put $G'=G[V\backslash\{v_0, y\}]$. Since G is not factorizable, G' is not factorizable. Since v_0 is not a vertex of G', G' is locally finite and then from Theorem A there is a finite subset S of $V\backslash\{v_0, y\}$ such that $C_1(V\backslash[S\cup\{v_0, y\}])\geqslant|S|+1$. From (1) the subgraph $G[V\backslash\{v_0\}]$ is factorizable and then (1.1) holds for this subgraph. It follows that we have $C_1(V\backslash[S\cup\{v_0, y\}])=|S|+1$, i.e. $y\in\bigcup\{S\subseteq V\backslash\{v_0\}\colon S \text{ finite}, C_1(V\backslash[S\cup\{v_0\}])=|S|\}$.

(3) *If* (1.1) *holds for G, then*

$$\bigcup\{S\subseteq V\backslash\{v_0\}\colon S \textit{ finite}, C_1(V\backslash[S\cup\{v_0\}])=|S|\}$$
$$\subseteq\bigcup\{S\subseteq V\backslash\{v_0\}\colon S \textit{ finite}, C_{cr}(V\backslash[S\cup\{v_0\}])=|S|\}.$$

This results clearly from Lemma 3.3 with $k=0$.

(4) *If* (1.1) *holds for G and if* (2.2) *does not hold for G, then G is not factorizable.*

Assume that G has a 1-factor F. Then there is $y\in V$ such that $\{v_0, y\}\in F$. Since (2.2) does not hold for G, there is a finite subset S of $V\backslash\{v_0\}$ such that $y\in S$ and

$C_{cr}(V\setminus[S\cup\{v_0\}])=|S|$. It follows that the subgraph $G[V\setminus\{v_0, y\}]$ does not satisfy (2.1), and then by Lemma 3.3 this subgraph does not satisfy (1.1). Therefore from Theorem A $G[V\setminus\{v_0, y\}]$ is not factorizable. On the other hand, since F is a 1-factor of G containing the edge $\{v_0, y\}$, $F\setminus\{v_0, y\}$ is a 1-factor of $G[V\setminus\{v_0, y\}]$, and the contradiction follows achieving the proof of (3).

(5) *If G is factorizable, then* (1.1) *holds for G.*

If G is factorizable and if S is a subset of V, every connected component of $G[V\setminus S]$ with odd cardinality is clearly joined to S by every 1-factor of G. Therefore (1.1) holds for G.

The proof of Theorem 3.1 and Corollary 3.2 is now complete. If G is factorizable, then by (4) and (5) conditions (1.1) and (2.2) hold. Therefore condition (1.2) holds by (3). From Lemma 3.3 with $k=0$ it follows that condition (2.1) holds.

From (2), (3) and (4) it follows that conditions (1.1) and (1.2)—or conditions (2.1) and (2.2)—are sufficient for the existence of a 1-factor of G.

References

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